Received: *November 6, 2001,* Accepted: *December 6, 2001*

Superconformal field theories for compact G_2 manifolds

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Abstract: We present the construction of exactly solvable superconformal field theories describing type-II string models compactified on compact G_2 manifolds. These models are defined by anti-holomorphic quotients of the form $(\mathrm{CY}\times S^1)/\mathbb{Z}_2$, where we realize the Calabi-Yau as a Gepner model. In the superconformal field theory the $\mathbb{Z}_2$ acts as charge conjugation implying that the representation theory of a $\mathcal{W}(2,4,6,8,10)$ algebra plays an important rôle in the construction of these models. Intriguingly, in all three examples we study, including the quintic, the massless spectrum in the $\mathbb{Z}_2$ twisted sector of the superconformal field theory differs from what one expects from the supergravity computation. This discrepancy is explained by the presence of a discrete NS-NS background two-form flux in the Gepner model.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua, Conformal Field Models in String Theory, Conformal and W Symmetry.

JHEP12(2001)006

Contents

1. Introduction

Recently, we have seen an intensified effort to reveal the structure of M-theory compactifications on manifolds with exceptional holonomy [1]–[13]. The main focus in recent developments was on non-compact examples of G_2 and Spin(7) holonomy where explicit metrics have been constructed [14]–[18]. Moreover, flop transitions on such manifolds allow for purely geometric M-theory lifts of the so-called Vafa duality [19]–[24]. An alternative lift of Vafa duality to M-theory has been considered in [25].

M-theory on compact seven dimensional G_2 manifolds are of special interest, as they lead to four dimensional effective theories with $\mathcal{N} = 1$ space-time supersymmetry. Since M-theory on smooth G_2 manifolds only gives rise to abelian gauge symmetries with non-chiral matter, it is clear that interesting phenomenology can only be realized on singular spaces. The kinds of singularities giving rise to non-abelian gauge symmetries and chiral matter have been analyzed in [9, 10, 11].

Most explicit compact G_2 manifolds constructed so far are given by certain toroidal orbifolds. This class includes both the models constructed by Joyce and the ones resulting from an M-theory lift of certain type-IIA orientifolds with D6-branes [26, 27]. Another large class of G_2 manifolds is supposed to result from anti-holomorphic $\mathbb{Z}_2$ quotients of Calabi-Yau manifolds times a circle. Phase transitions in the M-theory moduli space of such manifolds have been investigated in [4, 7, 12].

Since one is not equipped with a microscopic quantum M-theory, one can only study such models in the large radius limit where the supergravity approximation is valid. What one can do however is to compactify M-theory on a further S^1 down to three dimensions and employ the duality with type-IIA string theory, where computations in the small distance regime are in principle possible. For carrying out such a computation it is necessary to exactly solve the non-linear sigma model in this curved background. Even though, except for toroidal orbifolds, this is technically beyond our abilities, sometimes pure conformal field theory (CFT) considerations have proven to be successful in providing models which accidentally correspond to certain points in the deep interior of, for instance, the Calabi-Yau moduli space. The most prominent examples are certainly given by the so-called Gepner models [28, 29, 30], which use tensor products of minimal models of the $\mathcal{N} = 2$ super Virasoro algebra equipped by a GSO projection in the internal conformal field theory. Such superconformal field theories (SCFT) have been identified with certain points in the moduli space of Calabi-Yau threefolds given by Fermat type hypersurfaces in weighted projective spaces [29, 31, 32]. Other examples are given by so called $(0, 2)$ generalizations of the Gepner models which appear in heterotic compactifications with $\mathcal{N} = 1$ supersymmetry in four dimensions [33]. Moreover, Gepner models also served as a powerful tool in investigating stable BPS as well as non-BPS D-branes present in Calabi-Yau compactifications in the stringy regime [34, 35, 36].

In this paper we present a class of exactly solvable superconformal fields theories which are argued to correspond to certain points in the moduli space of type-II compactifications on G_2 manifolds. These manifolds are given by anti-holomorphic quotients of the form $(\mathrm{CY}\times S^1)/\mathbb{Z}_2$, where the Calabi-Yau manifold is given by a Fermat type hypersurface in a weighted projective space and is described in the SCFT by a Gepner model. Note that the general structure of the SCFT describing G_2 manifolds has been investigated in [37, 38, 39] but except toroidal orbifolds no explicit SCFT has been found so far. On the technical level we have to implement the anti-holomorphic $\mathbb{Z}_2$ action in the corresponding Gepner model, which turns out to be nothing else than conjugation of the U(1) charges in each factor theory. However, the determination of the action of charge conjugation on all states in the Hilbert space of a Gepner model is quite challenging and we present here the solution to this problem at least for the $k \in \{1, 2, 3, 6\}$ minimal models. We argue that the general solution to this problem is related to the so far unknown representation theory of a $\mathcal{W}(2, 4, 6, 8, 10)$ algebra.

Once the $\mathbb{Z}_2$ action is known on the entire Hilbert space, it is fairly straightforward to compute the orbifold partition function, including the new $\mathbb{Z}_2$ twisted sector, and to determine the massless spectrum in three space-time dimensions. For all models studied in this paper the massless spectrum disagrees with what one naively expects from the supergravity analysis. However, we will show that this is not surprising at all. It is known from the mirror symmetry analysis that the Gepner models correspond to points in Calabi-Yau moduli space with radii at the string scale and non-trivial background NS-NS two-form fluxes turned on. Under the anti-holomorphic involution the continuous moduli related to these fluxes are projected out, but nevertheless certain discrete values are still allowed. Therefore, the resolution of the puzzle stated above is simply that the supergravity model and the Gepner model occupy disconnected branches of the G_2 moduli space. Besides that, one expects world-sheet instanton corrections to be relevant in the stringy regime anyway which might lead to different phases with different massless modes.

This paper is organized as follows. In section 2 we briefly review the relevant aspects of the construction of Gepner models. In section 3 we determine the action of the anti-holomorphic involution for some of the $\mathcal{N} = 2$ unitary models, which allows us to study at least some of the Gepner models. In section 4, for the quotient of the quintic $\mathbb{P}_5[5]$, we explicitly compute the one-loop partition function and determine the massless spectrum including fields from the $\mathbb{Z}_2$ twisted sector. In section 5 we derive the expected geometric large radius result for the quintic and point out the discrepancy with the SCFT result. Section 6 provides more complicated SCFT examples involving also the $(k = 6)$ unitary model. In section 7 we present the computation of these G_2 compactifications in the supergravity limit and compare the results to the SCFT models. In addition we provide some material on the Cartan-Leray spectral sequence. Section 8 provides the resolution of the puzzle concerning the different results for the SCFT and supergravity computation. Finally, section 9 contains our conclusions and mentions some open problems.

2. Review of Gepner models

The goal of this paper is to study type-II compactifications on G_2 manifolds of the form

$$\frac{\mathrm{CY} \times S^1}{\sigma^*}, \tag{2.1}$$

where the Calabi-Yau threefold is given by a Fermat type hypersurface in a weighted

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projective space $\mathbb{P}_{w_1,w_2,w_3,w_4,w_5}[d]$ with $d = \sum w_i$. The anti-holomorphic involution acts on the homogeneous coordinates z_i of the projective space by complex conjugation and on the real coordinate, y, parameterizing the S^1 by a reflection. Going deep inside the Kähler moduli space of the Calabi-Yau there exists a point where the exact $\mathcal{N} = (2,2)$ superconformal field theory is explicitly known and described by a Gepner model [28]. In the following we work at this special point in moduli space. However, before continuing the construction of SCFTs for the G_2 manifolds (2.1), we need to review some aspects of Gepner's construction. For readers not familiar with Gepner models, we would like to refer them to the original literature [28].

2.1 $N = 2$ unitary models

Gepner models are given by tensor products of rational models of the $\mathcal{N} = 2$ super Virasoro algebra, so that the central charge of all tensor models adds up to $c = 9$. The $\mathcal{N} = 2$ extension of the Virasoro algebra contains besides the energy momentum tensor L two fermionic superpartners $G^{\pm}$ of conformal dimension $h = 3/2$ and one bosonic U(1) current j of conformal dimension $h = 1$. For future reference we give here the explicit form of the $\mathcal{N} = 2$ super Virasoro algebra

$$\begin{aligned}
[L_m, L_n] &= (m-n)\, L_{m+n} + \frac{c}{12}\, n(n^2-1)\, \delta_{m+n,0} \\
[L_m, j_n] &= -n\, j_{m+n} \\
[L_m, G_r^{\pm}] &= \left(\frac{m}{2} - r\right) G_{m+r}^{\pm} \\
[j_m, j_n] &= \frac{c}{3}\, n \delta_{m+n,0} \\
[j_m, G_r^{\pm}] &= \pm G_{m+r}^{\pm} \\
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)\, j_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right) \delta_{r+s,0} \\
\{G_r^+, G_s^+\} &= \{G_r^-, G_s^-\} = 0\,.
\end{aligned} \qquad (2.2)$$

The rational models are classified and the central charge is known to be restricted to the discrete series

$$c = \frac{3k}{k+2}\,, \qquad k \in \mathbb{Z}_+\,. \qquad (2.3)$$

For each level k there exists only a finite number of highest weight representations (h, q) labeled by their conformal dimension and their U(1) charge

$$\begin{aligned}
h &= \frac{l(l+2) - m^2}{4(k+2)} + \frac{s^2}{8} \\
q &= -\frac{m}{k+2} + \frac{s}{2}\,.
\end{aligned} \qquad (2.4)$$

The three indices (l, m, s) are restricted to lie in the standard range

$$\begin{aligned}
&0 \le l \le k, \\
&0 \le |m - s| \le l,
\end{aligned}$$

$$
\begin{aligned}
s &= \begin{cases} 0,2 & \text{NS sector} \\ \pm 1 & \text{R sector} \end{cases} \\
m &= m \bmod 2(k+2)\,, \quad s = s \bmod 4\,, \quad l+m+s = 0 \bmod 2\,.
\end{aligned} \tag{2.5}
$$

Note that all superconformal characters have been split into two bosonic pieces like for instance in the NS-sector

$$\chi_m^l = \chi_{m,s=0}^l + \chi_{m,s=2}^l\,. \tag{2.6}$$

These unitary models can be written as the product of the parafermions [40, 41] and a free U(1) current

$$\frac{\mathrm{SU}(2)_k}{\mathrm{U}(1)} \times \mathrm{U}(1)\,, \tag{2.7}$$

where the quotient of the $\mathrm{SU}(2)_k$ affine Lie algebra by the Cartan U(1) is precisely the parafermionic CFT and the additional U(1) can be identified with the abelian current in the $\mathcal{N}=2$ Virasoro algebra. The realization as a coset (2.7) enables one to easily determine the characters of the unitary representations of the $\mathcal{N}=2$ Virasoro algebra as branching functions [41, 28]

$$\chi_{m,s}^l(\tau) = \sum_{j=1}^{k} C_{m-(4j+s)}^l(\tau)\Theta_{2m-(k+2)(4j+s),2k(k+2)}\left(\tau, \frac{z}{k+2}\right). \tag{2.8}$$

The C_m^l are called string functions of $\mathrm{SU}(2)_k$ and are related to the characters κ_m^l of the parafermionic theory by

$$\kappa_m^l(\tau) = \eta(\tau)\, C_m^l(\tau)\,. \tag{2.9}$$

These string functions are explicitly known

$$C_m^l(\tau) = \eta(\tau)^{-3} \sum_{\substack{(x,y)\in\mathbb{R}^2 \\ -|x|<|y|\leq|x| \\ (x,y) \text{ or } (1/2-x,1/2+y)\in((l+1)/2(k+2),m/2k)+\mathbb{Z}^2}} \mathrm{sign}(x)\, q^{(k+2)x^2-ky^2} \tag{2.10}$$

with $q = e^{2\pi i\tau}$ and will play an important rôle in the following.

2.2 Gepner's construction

It had been known that in order to get $\mathcal{N}=2$ space-time supersymmetry in a four dimensional type-II compactification one needs $\mathcal{N}=(2,2)$ supersymmetry on the two-dimensional world-sheet [42]. Moreover, there must exist a spectral flow operator relating the NS and the R sector of the SCFT.

Gepner realized the internal SCFT describing the Calabi-Yau manifold by tensor products of unitary models of the $\mathcal{N}=2$ Virasoro algebra such that the central charges adds up to $c=9$. The remaining central charge of $c=3$ is occupied in light-cone gauge by two flat bosons X^μ with $\mu=2,3$ and their fermionic superpartners ψ^μ. The latter ones yield a realization of the $\mathrm{SO}(2)_1$ current algebra, namely in the NS sector the vacuum $(O_2)_{h=0,q=0}$ and the vector $(V_2)_{h=1/2,q=1}$ representation and in the R sector the spinor $(S_2)_{h=1/8,q=1/2}$

and the antispinor $(C_2)_{h=1/8,q=-1/2}$ representation. The characters for these representation read

$$\begin{aligned} O_2 &= \frac{1}{2}\left(\frac{\theta_3}{\eta}+\frac{\theta_4}{\eta}\right) \\ V_2 &= \frac{1}{2}\left(\frac{\theta_3}{\eta}-\frac{\theta_4}{\eta}\right) \\ S_2 &= \frac{1}{2}\left(\frac{\theta_2}{\eta}\right) \\ C_2 &= \frac{1}{2}\left(\frac{\theta_2}{\eta}\right). \end{aligned} \tag{2.11}$$

Neglecting in the following the flat space-time bosons, the starting point for the Gepner construction is the tensor product

$$\bigotimes_{i=1}^{N}(k_i)\times \mathrm{SO}(2)_1 \tag{2.12}$$

which contains highest weights denoted by

$$\prod_{i=1}^{N}(l_i,m_i,s_i)\times(\phi) \tag{2.13}$$

with $\phi\in\{O_2,V_2,S_2,C_2\}$.

In order to get a space-time supersymmetric string theory one has to implement a GSO projection, which in this case projects onto states with odd overall U(1) charge both in the left moving and the right moving sector. More formally, the GSO projection is realized by constructing a new modular invariant partition function utilizing the simple current[1] [43, 44]

$$J_{GSO}=(0,1,1)^N\otimes(C_2)\,. \tag{2.14}$$

The effect of this simple current construction is that it projects onto states with odd overall U(1) charge and arranges the surviving fields into orbits of finite length under the action of J_{GSO}. Besides the GSO projection one also has to make sure that from the individual factor theories only NS respectively R sector states are combined. This projection is implemented by the simple currents

$$J_a=\prod_{i=1}^{a-1}(0,0,0)\otimes(0,0,2)\otimes\prod_{i=a+1}^{N}(0,0,0)\otimes(V_2) \tag{2.15}$$

with $a=1,\ldots,N$. All these projections imply that in the vacuum orbit the massless states

$$\begin{aligned} &\left[(0,0,0)^N\otimes(V_2)\right]_L\times\left[(0,1,1)^N\otimes(C_2)\right]_R, \\ &\left[(0,1,1)^N\otimes(C_2)\right]_L\times\left[(0,0,0)^N\otimes(V_2)\right]_R \end{aligned} \tag{2.16}$$

and their charge conjugates survive, which are precisely the gravitinos of $\mathcal{N}=2$ space-time supersymmetry in four dimensions.

[1] More precisely, one first has to apply the bosonic string map exchanging SO(2) with $\mathrm{SO}(10)\times E_8$, apply the simple current techniques and finally map back to $\mathrm{SO}(2)_1$.

The massless spectrum of such a Gepner model can be determined from the modular invariant partition function. The vacuum orbit gives rise to the $\mathcal{N}=2$ supergravity multiplet in addition to one hypermultiplet containing the dilaton and the dualized NS-NS two form. Chiral states of the form

$$\left[\left(h=\frac{1}{2}, q=1\right)\otimes(O_2)\right]_L \times \left[\left(h=\frac{1}{2}, q=1\right)\otimes(O_2)\right]_R \qquad (2.17)$$

and their charge conjugates give rise to one hyper(vector)-multiplet, whereas anti-chiral states

$$\left[\left(h=\frac{1}{2}, q=1\right)\otimes(O_2)\right]_L \times \left[\left(h=\frac{1}{2}, q=-1\right)\otimes(O_2)\right]_R \qquad (2.18)$$

and their charge conjugates give rise to one vector(hyper)-multiplet in type-IIA (-IIB) string theory.

3. Anti-holomorphic involution

After we have reviewed the main ingredients of Gepner models in the last section we now move forward to the construction of SCFTs for G_2 manifolds. Starting with a Gepner model we also compactify the transversal boson X^3 on a circle of radius R so that in light-cone gauge we are left with only one non-compact direction X^2. The next step is to realize the anti-holomorphic involution σ^* in the Gepner model SCFT. Since formally the homogeneous coordinates Z_i can be identified with the chiral fields $(l_i, m_i, s_i) = (1,1,0)$ and the complex conjugates $\overline{Z}_i$ with $(l_i, m_i, s_i) = (1,-1,0)$, a natural ansatz is that charge conjugation in each individual factor theory of the SCFT is equivalent to complex conjugation in the corresponding geometry.

In the remainder of this section we study the action of this charge conjugation on states in the Hilbert space of one individual $\mathcal{N}=2$ unitary model. On the level of the $\mathcal{N}=2$ super Virasoro algebra (2.2) charge conjugation acts as

$$L_m \to L_m\,, \qquad j_m \to -j_m\,, \qquad G_r^+ \leftrightarrow G_r^- \qquad (3.1)$$

and is an automorphism of the algebra. The even generators are L and $\frac{1}{\sqrt{2}}(G^+ + G^-)$ which form the $\mathcal{N}=1$ superconformal algebra. Thus generically only an $\mathcal{N}=1$ superconformal symmetry survives the $\mathbb{Z}_2$ projection. Apparently, the states in the highest weight representation (HWR) (l,m,s) are mapped to states in the HWR $(l,-m,-s)$. Thus as long as $m \neq 0$ or $s \neq 0$ two states are simply exchanged under the action of σ^*. In particular, on all states in the R sector σ^* acts by exchange of two states.[2] However, for states in HWRs of the form $(l,0,0)$ the situation gets more complicated. Remember the form of the character in such a representation

$$\chi^l_{0,0}(\tau) = \sum_{j=1}^{k} C^l_{-4j}(\tau)\, \Theta_{-4(k+2)j,2k(k+2)}\left(\tau, \frac{z}{k+2}\right). \qquad (3.2)$$

[2]Actually, for k even $(l,m,s) = (l,(k+2)/2,1)$ are uncharged HWR in the R-sector, but since they have to combine with the spinor or antispinor representation of SO(2) they are all projected out by the GSO projection.

and that charge conjugation will map $\Theta_{m,2k(k+2)}$ to $\Theta_{-m,2k(k+2)}$. Therefore the involution might act non-trivially only on states contained in

$$C_0^l(\tau)\,\Theta_{0,2k(k+2)}(\tau)\,. \tag{3.3}$$

By the same argument as before only the single uncharged ground state in $\Theta_{0,2k(k+2)}$ is not mapped to a different state with opposite charge. Thus we conclude that σ^* can only act non-trivially on those states in the HWR $(l,0,0)$ which are counted in C_0^l. As expected these are precisely the neutral states in the HWR $(l,0,0)$. Since j is a free field, we can always factor out its contribution, which is precisely given by the Dedekind η-function

$$C_0^l(\tau) = \frac{1}{\eta(\tau)}\kappa_0^l(\tau)\,, \tag{3.4}$$

where κ_0^l are characters of HWRs of the parafermions. Unfortunately, the determination of the action of the anti-holomorphic involution on the neutral states contained in κ_0^l turns out to be a highly non-trivial task. But in a case by case study, we have managed to find a satisfactory solution at least for the four unitary models $k \in \{1,2,3,6\}$. In the following we present our analysis for these four cases separately.

3.1 Action of σ^* for the $(k=1)$ model

In this case the central charge is $c=1$ so that the parafermionic part is actually trivial, $\kappa_0^0 = 1$. The only interesting HWR is $(l,m,s)=(0,0,0)$. The fact that the parafermionic part is trivial means that all neutral states in this representation are generated by j_m alone. Said differently, there are so many null-states in this Verma module that all states containing modes L_m, $G_r^\pm$ can be expressed in terms of the modes j_m. The action of σ^* on j_m is known (3.1), so that for the trace over the HWR $(l,m,s)=(0,0,0)$ with an σ^* insertion we get

$$\chi_{0,0}^0(\sigma^*) = \mathrm{Tr}_{\mathcal{H}_{0,0}^0}\left(\sigma^* e^{2\pi i\tau L_0}\right) = \sqrt{\frac{2\eta}{\theta_2}}\,. \tag{3.5}$$

3.2 Action of σ^* for the $(k=2)$ model

In this case we have two interesting HWRs, $(l,m,s)=(0,0,0)$ and $(l,m,s)=(2,0,0)$. The central charge is $c=3/2$ so that the parafermions contribute $c=1/2$. It is a well known fact that the first non-trivial parafermionic model is identical to the Ising model, which can be considered as the first unitary model of the $\mathcal{N}=0$ Virasoro algebra. For the general parafermionic theory there exists a duality of coset models [45]

$$\frac{\mathrm{SU}(2)_k}{\mathrm{U}(1)} = \frac{\mathrm{SU}(k)_1\times\mathrm{SU}(k)_1}{\mathrm{SU}(k)_2}\,, \tag{3.6}$$

which means that the kth parafermionic model is identical to the first unitary model of the $\mathcal{WA}_{k-1}$ algebra. This $\mathcal{W}$ algebra is generated by primary fields of conformal dimension $\Delta\in\{2,3,\ldots,k\}$. Thus, naively one might expect that the chiral symmetry algebra of the parafermions contains infinitely many generators. However, as shown in [46] this is actually not true. All $\mathcal{WA}_k$ algebras truncate for the first unitary model, i.e. for $c=\frac{2(k-1)}{k+2}$, to a

$\mathcal{W}(2,3,4,5)$[3] algebra, which is different from the $\mathcal{WA}_4$ algebra. Moreover, this $\mathcal{W}(2,3,4,5)$ algebra truncates for the first three parafermionic models $k=2,3,4$ to the algebras $\mathcal{W}(2)$, $\mathcal{W}(2,3)$ and $\mathcal{W}(2,3,4)$, respectively. Please consult reference [46] for more details.

Back to the $(k=2)$ model, from the fact that the parafermionic model is contained in the unitary series of the Virasoro algebra we conclude that there is still a sufficient number of null-states in the Verma module that all uncharged states generated by $G_r^{\pm}$ can be expressed by j_m and L_m. Using that the $c=1/2$ theory is given by one free world-sheet fermion with its different spin structures we can write

$$\begin{aligned}
\chi^0_{0,0}(\sigma^*) &= \sqrt{\frac{2\eta}{\theta_2}}\,\frac{1}{2}\left(\sqrt{\frac{\theta_3}{\eta}}+\sqrt{\frac{\theta_4}{\eta}}\right)\\
\chi^2_{0,0}(\sigma^*) &= \sqrt{\frac{2\eta}{\theta_2}}\,\frac{1}{2}\left(\sqrt{\frac{\theta_3}{\eta}}-\sqrt{\frac{\theta_4}{\eta}}\right).
\end{aligned} \tag{3.7}$$

3.3 Action of σ^* for the $(k=3)$ model

Slowly the situation becomes more complicated. In this case the central charge is $c=9/5$ where the parafermions contribute $c=4/5$. This value for c is both contained in the unitary series of the Virasoro algebra and in the unitary series of the $\mathcal{WA}_2$ algebra and the model is known as the 3-states Potts model. The parafermionic respectively $\mathcal{WA}_2$ characters can be written in terms of characters of the $k=5$ Virasoro unitary model

$$\kappa^0_0 = \chi_0+\chi_3\,, \qquad \kappa^2_0=\chi_{\frac{2}{5}}+\chi_{\frac{7}{5}}\,, \tag{3.8}$$

where the indices denote the conformal dimensions of the HWR. Please consult appendix A for some basis data of the representation theory of the Virasoro algebra. The interpretation of κ^0_0 in terms of the generators of the $\mathcal{N}=2$ Virasoro algebra is as follows. Now, not the entire Hilbert space is generated by j_m and L_m alone. Instead at conformal dimension three a new field, W_3, appears. However, the normal ordered product of this field with itself gives rise to a null-state. The precise form of this field in terms of the generators $j, L, G^{\pm}$ can be found in [46]. What is important for us is that the new mode at level three can only be $G^+_{-\frac{3}{2}}G^-_{-\frac{3}{2}}|0\rangle$. Since the $G^{\pm}$ anticommute, under charge conjugation this state picks up a minus sign so that the action of σ^* on the HWRs $(l,m,s)=(0,0,0)$ and $(l,m,s)=(2,0,0)$ reads

$$\begin{aligned}
\chi^0_{0,0}(\sigma^*) &= \sqrt{\frac{2\eta}{\theta_2}}\,(\chi_0-\chi_3)\\
\chi^2_{0,0}(\sigma^*) &= \sqrt{\frac{2\eta}{\theta_2}}\,\left(\chi_{\frac{2}{5}}-\chi_{\frac{7}{5}}\right).
\end{aligned} \tag{3.9}$$

Note that this result is not obvious from the very beginning and we really needed to perform a quite detailed analysis of the uncharged states.

[3] We denote a $\mathcal{W}$-algebra with generators of conformal dimension $h\in\{2,\Delta_1,\ldots,\Delta_n\}$ as $\mathcal{W}(2,\Delta_1,\ldots,\Delta_n)$.

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That we found such a simple answer was only possible due to the decomposition of the parafermionic character in terms of Virasoro characters. The results presented in section 4 will provide a highly nontrivial consistency check for the correctness of (3.9). In particular, after a modular S transformation of (3.9) one obtains the σ^* twisted sector. This sector first must allow the interpretation as a partition function, second must satisfy level matching and third must be free of any tachyonic states.

For $k \geq 4$ the parafermionic part is not any longer a unitary model of the Virasoro algebra and we do not know in general the split of the parafermionic characters κ_0^l in σ^* even and σ^* odd parts. The only other model where we succeeded in finding this decomposition is the $k = 6$ unitary model.

3.4 Action of σ^* for the ($k = 6$) model

For $k = 6$ the central charge is $c = 9/4$ so that the parafermionic theory contributes $c = 5/4$. Even though this number is not contained in the unitary series of the $\mathcal{N} = 0$ Virasoro algebra it is a member of the unitary series of the $\mathcal{N} = 1$ super Virasoro algebra at $m = 6$

$$c = \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right). \tag{3.10}$$

Some inspection reveals that the vacuum character of the $k = 6$ parafermionic theory can be written as

$$\kappa_0^0 = \frac{1}{2}\left(\chi_0^{NS} + \chi_0^{\widetilde{NS}} + \chi_3^{NS} + \chi_3^{\widetilde{NS}}\right), \tag{3.11}$$

where again the lower indices denote the conformal dimensions of the HWRs. Expanding the superconformal characters we find for the first eight levels

$$\begin{aligned}
\frac{1}{2}\left(\chi_0^{NS} + \chi_0^{\widetilde{NS}}\right) &= q^{-\frac{5}{96}}\left(1 + q^2 + q^3 + 3q^4 + 3q^5 + 7q^6 + 8q^7 + 15q^8 + \cdots\right) \\
\frac{1}{2}\left(\chi_3^{NS} + \chi_3^{\widetilde{NS}}\right) &= q^{-\frac{5}{96}}\left(q^3 + q^4 + 3q^5 + 4q^6 + 7q^7 + 10q^8 + \cdots\right).
\end{aligned} \tag{3.12}$$

Thinking of these states as being generated by L_m and $G_r^{\pm}$ the number of states at each mass level in the first row of (3.12) agrees with the expected number of σ^* even states, whereas the number of states in the second row completely agrees with the expected number of σ^* odd states. Thus we claim that this pattern will continue to all mass levels and that the decomposition of the parafermionic vacuum character in terms of $\mathcal{N} = 1$ super Virasoro characters automatically reflects the desired split into σ^* even and σ^* odd states. The remaining κ_0^l characters have a similar decomposition

$$\begin{aligned}
\kappa_0^2 &= \frac{1}{2}\left(\chi_{\frac{1}{4}}^{NS} + \chi_{\frac{1}{4}}^{\widetilde{NS}} + \chi_{\frac{5}{4}}^{NS} + \chi_{\frac{5}{4}}^{\widetilde{NS}}\right) \\
\kappa_0^4 &= \frac{1}{2}\left(\chi_{\frac{1}{4}}^{NS} - \chi_{\frac{1}{4}}^{\widetilde{NS}} + \chi_{\frac{5}{4}}^{NS} - \chi_{\frac{5}{4}}^{\widetilde{NS}}\right) \\
\kappa_0^6 &= \frac{1}{2}\left(\chi_0^{NS} - \chi_0^{\widetilde{NS}} + \chi_3^{NS} - \chi_3^{\widetilde{NS}}\right).
\end{aligned} \tag{3.13}$$

Analogously to the vacuum character we conjecture that the action of charge conjugation on these characters is simply

$$\begin{aligned}
\chi^0_{0,0}(\sigma^*) &= \sqrt{\frac{2\eta}{\theta_2}}\frac{1}{2}\left[\left(\chi^{NS}_0-\chi^{NS}_3\right)+\left(\chi^{\widetilde{NS}}_0-\chi^{\widetilde{NS}}_3\right)\right] \\
\chi^2_{0,0}(\sigma^*) &= \sqrt{\frac{2\eta}{\theta_2}}\frac{1}{2}\left[\left(\chi^{NS}_{\frac{1}{4}}-\chi^{NS}_{\frac{5}{4}}\right)+\left(\chi^{\widetilde{NS}}_{\frac{1}{4}}-\chi^{\widetilde{NS}}_{\frac{5}{4}}\right)\right] \\
\chi^4_{0,0}(\sigma^*) &= \sqrt{\frac{2\eta}{\theta_2}}\frac{1}{2}\left[\left(\chi^{NS}_{\frac{1}{4}}-\chi^{NS}_{\frac{5}{4}}\right)-\left(\chi^{\widetilde{NS}}_{\frac{1}{4}}-\chi^{\widetilde{NS}}_{\frac{5}{4}}\right)\right] \\
\chi^6_{0,0}(\sigma^*) &= \sqrt{\frac{2\eta}{\theta_2}}\frac{1}{2}\left[\left(\chi^{NS}_0-\chi^{NS}_3\right)-\left(\chi^{\widetilde{NS}}_0-\chi^{\widetilde{NS}}_3\right)\right]. \qquad (3.14)
\end{aligned}$$

In section 6 we will see that this guess is supported by the consistency of the results we will obtain for the $\mathbb{Z}_2$ twisted sectors of the $(6)^4$ and the $(2)^3(6)^2$ Gepner models. The results derived in this section will provide the main technical information we need in order to construct the complete partition functions of the anti-holomorphic orbifold models.

3.5 Speculations about the σ^* action at arbitrary level

Unfortunately, we have not managed yet to find the decomposition of the uncharged characters into σ^* even and σ^* odd parts for generic level k. Apparently, this a pure CFT problem. In [46] the $\mathcal{W}$-algebra for the σ^* even part of the parafermionic CFT was determined to be a $\mathcal{W}(2,4,6,8,10)$ algebra. The structure constants of this $\mathcal{W}$-algebra are not known explicitly. For $c = 4/5$ we expect this $\mathcal{W}$ algebra to truncate to the Virasoro algebra and for $c = 5/4$ we expect it to truncate to the $\mathcal{W}(2,4,6)_3$ algebra, which is the bosonic projection of the $\mathcal{N} = 1$ Virasoro algebra.

Furthermore, the $\mathcal{W}(2,4,6,8,10)$ algebra should admit rational models for the unitary series of the parafermions $c = \frac{2(k-1)}{k+2}$, where in addition for all these values of c a HWR of conformal dimension $h = 3$ should appear. Then the parafermionic vacuum character would split like

$$\kappa^0_0 = \chi_0 + \chi_3 \qquad (3.15)$$

and the action of σ^* would simply be

$$\kappa^0_0(\sigma^*) = \chi_0 - \chi_3\,. \qquad (3.16)$$

Classifying the minimal models of this $\mathcal{W}(2,4,6,8,10)$-algebra and derive their modular transformation properties would be the main task on the way to the computational exploration of hundreds of different Gepner models.

4. SCFT for the $(\mathbb{P}_4[5]\times S^1)/\mathbb{Z}_2$ model

Since we are equipped with the action of σ^* on all states in the Hilbert space of the $k = 3$ unitary model, we are in the position to compute the one-loop partition function for the Gepner type model defined by

$$\frac{(3)^5\times S^1}{\sigma^*}, \qquad (4.1)$$

which is expected to geometrically correspond to the G_2 manifold

$$\frac{\mathbb{P}_4[5]\times S^1}{\sigma^*}\,. \tag{4.2}$$

Remember, that σ^* acts by charge conjugation on the $\mathcal{N}=2$ unitary factor models and by inversion $y\to -y$ on the coordinate compactified on a circle.

In the following we will demonstrate the construction of modular invariant partition functions for G_2 manifolds on this specific examples. The generalization to other Gepner models is straightforward.

As in section 3 we are now considering the Gepner model defined by the tensor product of five copies of the $k=3$ unitary model and two free fermions forming the $\mathrm{SO}(2)_1$ current algebra.

4.1 The free fermion part

Since we now compactify one more direction on a circle, we first have to decompose the $\mathrm{SO}(2)_1$ representations into $\mathrm{SO}(1)_1\times \mathrm{SO}(1)_1$ representations, where by $\mathrm{SO}(1)_1$ we mean the CFT of one free fermion, i.e. the Ising model. The first $\mathrm{SO}(1)_1$ factor corresponds to the flat direction and the second $\mathrm{SO}(1)_1$ factor to the direction compactified on S^1. The characters of the Ising model are

$$\begin{aligned} O_1 &= \frac{1}{2}\left(\sqrt{\frac{\theta_3}{\eta}}+\sqrt{\frac{\theta_4}{\eta}}\right)\\ V_1 &= \frac{1}{2}\left(\sqrt{\frac{\theta_3}{\eta}}-\sqrt{\frac{\theta_4}{\eta}}\right)\\ S_1 &= \sqrt{\frac{\theta_2}{2\eta}}. \end{aligned} \tag{4.3}$$

Starting with the $\mathrm{SO}(1)_1\times \mathrm{SO}(1)_1$ theory, the characters of the $\mathrm{SO}(2)_1$ CFT are given by taking orbits under the simple current $J=(V_1\,V_1)$

$$\begin{aligned} O_2 &= O_1\,O_1+V_1\,V_1\\ V_2 &= O_1\,V_1+V_1\,O_1\\ S_2 &= S_1\,S_1\\ C_2 &= S_1\,S_1\,. \end{aligned} \tag{4.4}$$

Due to the fusion rule $S_1\times V_1=S_1$ the spinor representation S_1 with conformal dimension $h=1/16$ is a fixed point under the action of the simple current and therefore $(S_1\,S_1)$ gives rise to the representations S_2 and C_2. Under a modular $S:\tau\to -1/\tau$ transformation the three representations (O_1,V_1,S_1) of the free fermion CFT transform as

$$S=\frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2}\\ 1 & 1 & -\sqrt{2}\\ \sqrt{2} & -\sqrt{2} & 0\end{pmatrix}. \tag{4.5}$$

Note that under the action of σ^* the characters O_1 is invariant, whereas V_1 is mapped to $-V_1$. Additionally, σ^* exchanges the spinor S_2 representation with the anti-spinor C_2 representation, so that it also acts as charge conjugation in the free fermion part of the CFT.

4.2 Space-time supersymmetry

We expect that under the action of σ^* half of the supersymmetry is broken, so that we are left with $N = 2$ supersymmetry in three dimensions. Indeed the gravitinos from equation (2.16) split into

$$\begin{aligned}&\left[(0,0,0)^5 \otimes (V_1O_1 + O_1V_1)\right]_L \times \left[(0,1,1)^5 \otimes (C_2) + (3,4,1)^5 \otimes (S_2)\right]_R ,\\ &\left[(0,1,1)^5 \otimes (C_2) + (3,4,1)^5 \otimes (S_2)\right]_L \times \left[(0,0,0)^5 \otimes (V_1O_1 + O_1V_1)\right]_R\end{aligned} \tag{4.6}$$

and are mapped under σ^* to

$$\begin{aligned}&\left[(0,0,0)^5 \otimes (-V_1O_1 + O_1V_1)\right]_L \times \left[(3,4,1)^5 \otimes (S_2) + (0,1,1)^5 \otimes (C_2)\right]_R ,\\ &\left[(3,4,1)^5 \otimes (S_2) + (0,1,1)^5 \otimes (C_2)\right]_L \times \left[(0,0,0)^5 \otimes (-V_1O_1 + O_1V_1)\right]_R\end{aligned} \tag{4.7}$$

so that precisely half of the supercharges are projected out. After the σ^* projection the vacuum orbit contributes the three dimensional $\mathcal{N} = 2$ supergravity multiplet in addition to one chiral multiplet.

Moreover, the states $[(3,3,0)^5 \otimes (O_1\, O_1)]_{L,R}$ of conformal dimension $(h,q) = (3/2,3)$ are mapped under σ^* to their charge conjugate states $[(3,-3,0)^5 \otimes (O_1\, O_1)]_{L,R}$ so that the symmetric linear combinations survive. Note that these linear combinations correspond to the covariantly constant three-form on the G_2 manifold.

4.3 Massless states in the $\mathbb{Z}_2$ untwisted sector

The three dimensional massless spectrum appearing in the $\mathbb{Z}_2$ untwisted sector is quite general. In fact the σ^* action on massless chiral or anti-chiral states exchanges

$$\left[\left(h = \frac{1}{2}, q = 1\right) \otimes (O_1O_1 + V_1V_1)\right]_L \times \left[\left(h = \frac{1}{2}, q = \pm 1\right) \otimes (O_1O_1 + V_1V_1)\right]_R \tag{4.8}$$

with its charge conjugate

$$\left[\left(h = \frac{1}{2}, q = -1\right) \otimes (O_1O_1 - V_1V_1)\right]_L \times \left[\left(h = \frac{1}{2}, q = \mp 1\right) \otimes (O_1O_1 - V_1V_1)\right]_R \tag{4.9}$$

and therefore leads to one chiral multiplet for both the type-IIA and the type-IIB string. Thus, from the untwisted sector we get the supergravity multiplet and $h_{11} + h_{21} + 1$ chiral multiplets. This is in agreement with the result from the geometric computation where the number of chiral multiplets is given by $b_2 + b_3 = h_{11} + h_{21} + 1$. Since the uncharged states are not touched at all, performing the mirror sign flip $\mathrm{U}(1)_R \to -\mathrm{U}(1)_R$ in the $c = 9$ Gepner model does lead to isomorphic SCFTs after the $\mathbb{Z}_2$ orbifold, as well.

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4.4 Partition function in the $\mathbb{Z}_2$ twisted sector

The recipe for finding the partition function in the $\mathbb{Z}_2$ twisted sector is to first compute the trace with the σ^* insertion $\sigma^* \underset{1}{\square}$ and then apply a modular S-transformation to get the sector $1 \underset{\sigma^*}{\square}$. For the $1 \underset{\sigma^*}{\square}$ sector a number of non-trivial consistency conditions arise. First, this sector must be level matched which means that all twisted states must satisfy

$$h_L - h_R \in \mathbb{Z}/2\,. \tag{4.10}$$

Second, the twisted sector must really admit the interpretation as a partition function with non-negative integer coefficient. Third, the twisted sector must vanish and there must be the same number of space-time bosons and fermions. Finally, there must not be any tachyons in the model. The partition function in the $\mathbb{Z}_2$ twisted sector is then

$$\frac{1}{2}\left(1 \underset{\sigma^*}{\square} + \sigma^* \underset{\sigma^*}{\square}\right) \tag{4.11}$$

where $\sigma^* \underset{\sigma^*}{\square}$ can be obtained from $1 \underset{\sigma^*}{\square}$ by applying a modular $T : \tau \to \tau + 1$ transformation.

As we argued in section 3 only uncharged states, counted in the $\chi^l_{0,0}$ characters for each factor theory, can contribute to the $\sigma^* \underset{1}{\square}$ sector. For $k = 3$ there are only two uncharged characters, $\chi^0_{0,0}$ and $\chi^2_{0,0}$, which appear in different orbits under the GSO simple current (2.14). Note that all these uncharged states are combined with the vector representation, V_2 of SO(2). Moreover, only the trivial Kaluza-Klein and winding mode for the compact S^1 direction is invariant under σ^*, so that collecting everything together we obtain the following partition function

$$\sigma^* \underset{1}{\square} = \frac{2}{|\eta|^2}\left|\sqrt{\frac{\eta}{\theta_2}}\right|^2 |O_1 V_1 - V_1 O_1|^2 \sum_{n=0}^{5}\binom{5}{n}\left|\left(\chi^0_{0,0}(\sigma^*)\right)^n\,\left(\chi^2_{0,0}(\sigma^*)\right)^{5-n}\right|^2. \tag{4.12}$$

For the free fermion part we are carefully treating the order of the two SO(1) factors, whereas for simplicity for the five $k = 3$ tensor factors we introduced a combinatorial degeneracy. Using equation (3.9) yields

$$\sigma^* \underset{1}{\square} = \frac{2^6}{|\eta|^2}\left|\sqrt{\frac{\eta}{\theta_2}}\right|^{12} |O_1 V_1 - V_1 O_1|^2 \sum_{n=0}^{5}\binom{5}{n}\left|(\chi_0 - \chi_3)^n\,\left(\chi_{\frac{2}{5}} - \chi_{\frac{7}{5}}\right)^{5-n}\right|^2. \tag{4.13}$$

The next step is to apply a modular S transformation. For the free fermions we obtain

$$(O_1 V_1 - V_1 O_1) \to \frac{1}{\sqrt{2}}(S_1(O_1 + V_1) - (O_1 + V_1)S_1) \tag{4.14}$$

which apparently vanishes and also guarantees that the number of space-time bosons contained in the first term, $S_1(O_1 + V_1)$, agrees with the number of space-time fermions counted in the second term $(O_1 + V_1)S_1$. Using the formulae collected in appendix A, under a modular S-transformation the characters of the three states Potts model behave as follows

$$\begin{aligned}
(\chi_0 - \chi_3) &\to \sqrt{\frac{4}{5}}\left(\sin\left(\frac{\pi}{5}\right)(\chi_{\frac{1}{8}} + \chi_{\frac{13}{8}}) + \sin\left(\frac{3\pi}{5}\right)(\chi_{\frac{1}{40}} + \chi_{\frac{21}{40}})\right)\\
\left(\chi_{\frac{2}{5}} - \chi_{\frac{7}{5}}\right) &\to \sqrt{\frac{4}{5}}\left(-\sin\left(\frac{3\pi}{5}\right)(\chi_{\frac{1}{8}} + \chi_{\frac{13}{8}}) + \sin\left(\frac{\pi}{5}\right)(\chi_{\frac{1}{40}} + \chi_{\frac{21}{40}})\right).
\end{aligned} \tag{4.15}$$

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Thus, for the twisted sector partition function we get an expression of the form

$$1\underset{\sigma^*}{\square} = \frac{2^5}{|\eta|^2}\left|\sqrt{\frac{\eta}{\theta_4}}\right|^{12} |S_1(O_1+V_1)-(O_1+V_1)S_1|^2 \times$$
$$\times \sum_{n=0}^{5}\binom{5}{n}\left|\left(\chi_{\frac{1}{40}}+\chi_{\frac{21}{40}}\right)^n\left(\chi_{\frac{1}{8}}+\chi_{\frac{13}{8}}\right)^{5-n}\right|^2 . \tag{4.16}$$

Note that all terms in the sum satisfy the level matching condition

$$h_L - h_R \in \frac{\mathbb{Z}}{2}, \tag{4.17}$$

which we consider as a highly non-trivial check that our σ^* action in the $k=3$ unitary model is indeed correct. Moreover, the computed twisted sector partition function really has the interpretation as a trace over states in a new sector of the Hilbert space. All the requirements we mentioned at the beginning of this subsection are therefore satisfied.

Since the sum in (4.16) starts like

$$\left|\left(\chi_{\frac{1}{40}}+\chi_{\frac{21}{40}}\right)^5\right|^2 + \text{higher terms} \tag{4.18}$$

the ground state energy in the σ^* twisted sector is

$$E = \frac{1}{16} \tag{4.19}$$

and there do not appear any new massless states. Thus, the overall massless spectrum of the $\frac{(3)^5\times S^1}{\sigma^*}$ model is

$$\text{SUGRA} + 103_u \text{ chiral multiplets}. \tag{4.20}$$

We conclude that at the Gepner point of the quintic σ^* acts freely, which has to be compared with what happens in the large radius limit where we can describe the model geometrically.

5. Geometric interpretation of the $(\mathbb{P}_4[5]\times S^1)/\mathbb{Z}_2$ model

5.1 The geometry

The antiholomorphic involution on the Calabi-Yau manifold is induced from the ambient projective space. Since the different $k=3$ factors are not interchanged by the orbifold the involution is the obvious $z_i \to \bar{z}_i$ (the A-type involution in the language of [4]). The Fermat quintic

$$\{z_0^5+z_1^5+z_2^5+z_3^5+z_4^5=0\}\in\mathbb{P}_{1,1,1,1,1} \tag{5.1}$$

is obviously mapped to itself by the involution.

Now we want to determine the de Rahm cohomology of the quotient, and for this we have to find the invariant classes on the initial space $\mathbb{P}_4[5]\times S^1$. But the classes on a Cartesian product are simply the product of the cohomology classes and it suffices to discuss the forms on the two factors separately:

$\mathrm{d}y \in H^1(S^1)$ The volume form on the S^1 is odd under σ^*.

$\omega \in H^{1,1}(\mathbb{P}_4[5])$	The Kähler form is induced from the ambient space. Therefore it is odd under the antiholomorphic involution since the Kähler form on $\mathbb{P}_4$ is odd. (In our examples this will always be the case for all h^{11} classes)
$\Omega, \bar{\Omega} \in H^{3,0} \oplus H^{0,3}$	The involution exchanges $H^{3,0}$ and $H^{0,3}$, so there is one invariant and one antiinvariant combination. We choose the phase of Ω such that $\mathrm{Re}(\Omega)$ is invariant.
$\eta_i \in H^{2,1} \oplus H^{1,2}$	By the same argument $h^{21} = 101$ of the $2h^{21}$ forms are even, say η_i with $i = 1, \ldots, h^{21}$.

The nontrivial Betti numbers of the quotient $(\mathbb{P}_4[5] \times S^1)/\sigma^*$ are then

$b^1 = 0$	since $\mathrm{d}y$ is projected out. This is necessary for G_2 holonomy.
$b^2 = 0$	since there are no invariant 2-forms.
$b^3 = 103$	from the 101 invariant forms η_i, the form $\mathrm{d}y \wedge \omega$ and from $\mathrm{Re}(\Omega)$.

5.2 Singularities

Away from the fixed points of the σ^* action the quotient is a manifold. But the geometric group action has fixed loci that give rise to A_1 singularities. The fixed points on $\mathbb{P}_4[5] \times S^1$ are the product of the individual fixed point loci. Let $\Sigma \in \mathbb{P}_4[5]$ be the real points of the quintic, then the overall fixed set is $(\Sigma \times \{0\}) \cup (\Sigma \times \{1/2\})$.

Σ is the solution of the real polynomial equation $x_1^5 + \cdots + x_5^5 = 0$ in $\mathbb{RP}_4$. We want to determine its topology: First note that $\mathbb{R} \to \mathbb{R}, x \mapsto x^5$ is bijective, so we might just as well analyze $x_1 + \cdots + x_5 = 0$ in $\mathbb{RP}_4$. On the double cover S^4 of $\mathbb{RP}_4$ this equation determines an equatorial S^3 (the intersection of one 4-plane with the unit S^4 embedded in $\mathbb{R}^5$). Finally, modding out the remaining (antipodal) $\mathbb{Z}_2$-action leads to $\Sigma \simeq \mathbb{RP}_3$. Locally the involution acts as $(\xi_1, \xi_2, \xi_3, y) \to (\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, -y)$. So it leaves $\mathrm{Re}\,\xi_1, \mathrm{Re}\,\xi_2, \mathrm{Re}\,\xi_3$ invariant and inverts the 4 other directions. So the normal direction over the fixed set looks like $\mathbb{R}^4/\mathbb{Z}_2$.

We have characterized the singularities as two disjoint copies of $\mathbb{RP}_3$ with A_1 normal bundle. Can one resolve these singularities? It is anticipated by [1, 7] that this is not possible within G_2 holonomy, but no proof is available. However the following argument shows that the usual resolution (gluing Eguchi-Hansen spaces for the A_1-singularities) is not possible.

The Eguchi-Hansen space contains a nontrivial S^2 whose size must become a modulus of the resulting G_2 manifold. But the moduli space of G_2 metrics is b_3-dimensional, so the resolution must increase b_3. This can only happen if the singular locus has $b_1 > 0$ because the S^2 has to combine with a 1-cycle in the base to form a 3-cycle. But the $\mathbb{RP}_3$ has $b_1 = 0$. One might hope that a flop (see [20]) might ameliorate the situation but this will not help in the present case: it would only exchange $\mathbb{RP}_3 \times \mathbb{R}^4/\mathbb{Z}_2 \longleftrightarrow \mathbb{R}^4/\mathbb{Z}_2 \times \mathbb{RP}_3$

Nevertheless, applying an adiabatic argument we expect precisely one massless chiral multiplet from the local geometry around the singularity. This multiplet arises from dimensionally reducing the six-dimensional abelian gauge field living at the orbifold fixed point of $\mathbb{R}^4/\mathbb{Z}_2$ on $\mathbb{RP}_3$. Thus, there is a clear mismatch between the CFT and the geometric computation. The other examples studied in the next section feature a similar discrepancy of the twisted sector massless modes. We will resolve this puzzle after we present two more SCFT examples.

6. SCFTs for the $(\mathbb{P}_{1,1,1,1,4}[8]\times S^1)/\mathbb{Z}_2$ and $(\mathbb{P}_{1,1,2,2,2}[8]\times S^1)/\mathbb{Z}_2$ model

Since, we know the action of σ^* for the $k=6$ unitary model, the two models

$$\frac{(6)^4\times S^1}{\sigma^*}\,,\qquad \frac{(2)^3\,(6)^2\times S^1}{\sigma^*} \tag{6.1}$$

are also calculable. Geometrically, these two Gepner models correspond to the G_2 manifolds

$$\frac{\mathbb{P}_{1,1,1,1,4}[8]_{1,149}\times S^1}{\sigma^*}\,,\qquad \frac{\mathbb{P}_{1,1,2,2,2}[8]_{2,86}\times S^1}{\sigma^*}\,, \tag{6.2}$$

where the index denotes the Hodge numbers (h_{11},h_{21}).

6.1 $(\mathbb{P}_{1,1,1,1,4}[8]\times S^1)/\mathbb{Z}_2$

In complete analogy to the quintic the massless spectrum in the $\mathbb{Z}_2$ untwisted sector consists of the $\mathcal{N}=2$ supergravity multiplet in addition to 151 chiral multiplets. In the ${\scriptstyle\sigma^*}\underset{1}{\square}$ sector the four uncharged characters $\chi^{2j}_{0,0}$ for $j\in\{0,1,2,3\}$ can appear. In contrast to the $k=3$ Gepner model here the pairs of characters $(\chi^0_{0,0},\chi^6_{0,0})$ and $(\chi^2_{0,0},\chi^4_{0,0})$ appear in the same orbit under the spectral flow. Therefore the ${\scriptstyle\sigma^*}\underset{1}{\square}$ partition function reads

$$\begin{aligned}{\scriptstyle\sigma^*}\underset{1}{\square} = \frac{2}{|\eta|^2}\left|\sqrt{\frac{\eta}{\theta_2}}\right|^2 |O_1V_1-V_1O_1|^2\frac{1}{2}\sum_{i,j,k,l=0}^{3}\Big|&\chi^{2i}(\sigma^*)\chi^{2j}(\sigma^*)\chi^{2k}(\sigma^*)\chi^{2l}(\sigma^*)+\\ &+\chi^{6-2i}(\sigma^*)\chi^{6-2j}(\sigma^*)\chi^{6-2k}(\sigma^*)\chi^{6-2l}(\sigma^*)\Big|^2.\end{aligned} \tag{6.3}$$

Inserting (3.14) and using the modular transformation properties summarized in appendix B

$$\begin{aligned}
(\chi^{NS}_0-\chi^{NS}_3) &\rightarrow \frac{1}{\sqrt{2}}(\chi^{NS}_{\frac{1}{32}}+\chi^{NS}_{\frac{33}{32}})+\chi^{NS}_{\frac{5}{32}}\\
(\chi^{NS}_{\frac14}-\chi^{NS}_{\frac54}) &\rightarrow \frac{1}{\sqrt{2}}(\chi^{NS}_{\frac{1}{32}}+\chi^{NS}_{\frac{33}{32}})-\chi^{NS}_{\frac{5}{32}}\\
(\chi^{\widetilde{NS}}_0-\chi^{\widetilde{NS}}_3) &\rightarrow c_1(\chi^R_{\frac{5}{16}}+\chi^R_{\frac{29}{16}})+c_2(\chi^R_{\frac{1}{16}}+\chi^R_{\frac{9}{16}})\\
(\chi^{\widetilde{NS}}_{\frac14}-\chi^{\widetilde{NS}}_{\frac54}) &\rightarrow -c_2(\chi^R_{\frac{5}{16}}+\chi^R_{\frac{29}{16}})+c_1(\chi^R_{\frac{1}{16}}+\chi^R_{\frac{9}{16}})
\end{aligned} \tag{6.4}$$

with

$$c_1=\frac{1}{\sqrt{6}}\left(\cos\left(\tfrac{5\pi}{24}\right)-\cos\left(\frac{11\pi}{24}\right)\right),\qquad c_2=\frac{1}{\sqrt{6}}\left(\cos\left(\tfrac{\pi}{24}\right)+\cos\left(\frac{7\pi}{24}\right)\right). \tag{6.5}$$

we obtain for the twisted sector partition function

$$1\underset{\sigma^*}{\square}=\frac{2^4}{|\eta|^2}\left|\sqrt{\frac{\eta}{\theta_4}}\right|^{10}|S_1(O_1+V_1)-(O_1+V_1)S_1|^2\,I(q,\overline{q})\,. \tag{6.6}$$

with

$$
\begin{aligned}
I(q,\overline{q}) = & \frac{1}{128}\left|(\chi^R_{\frac{1}{16}}+\chi^R_{\frac{9}{16}})^4\right|^2+\frac{1}{8}\left|(\chi^{NS}_{\frac{1}{32}}+\chi^{NS}_{\frac{33}{32}})^4\right|^2+\frac{3}{16}\left|(\chi^R_{\frac{1}{16}}+\chi^R_{\frac{9}{16}})^2(\chi^{NS}_{\frac{1}{32}}+\chi^{NS}_{\frac{33}{32}})^2\right|^2+\\
&+\frac{1}{32}\left|(\chi^R_{\frac{1}{16}}+\chi^R_{\frac{9}{16}})^3(\chi^R_{\frac{5}{16}}+\chi^R_{\frac{29}{16}})\right|^2+\frac{3}{8}\left|(\chi^R_{\frac{1}{16}}+\chi^R_{\frac{9}{16}})(\chi^{NS}_{\frac{1}{32}}+\chi^{NS}_{\frac{33}{32}})^2(\chi^R_{\frac{5}{16}}+\chi^R_{\frac{29}{16}})\right|^2+\\
&+\frac{3}{64}\left|(\chi^R_{\frac{1}{16}}+\chi^R_{\frac{9}{16}})^2(\chi^R_{\frac{5}{16}}+\chi^R_{\frac{29}{16}})^2\right|^2+\frac{3}{16}\left|(\chi^{NS}_{\frac{1}{32}}+\chi^{NS}_{\frac{33}{32}})^2(\chi^R_{\frac{5}{16}}+\chi^R_{\frac{29}{16}})^2\right|^2+\\
&+\frac{1}{32}\left|(\chi^R_{\frac{1}{16}}+\chi^R_{\frac{9}{16}})(\chi^R_{\frac{5}{16}}+\chi^R_{\frac{29}{16}})^3\right|^2+\frac{1}{128}\left|(\chi^R_{\frac{5}{16}}+\chi^R_{\frac{29}{16}})^4\right|^2+\\
&+\frac{3}{4}\left|(\chi^R_{\frac{1}{16}}+\chi^R_{\frac{9}{16}})^2(\chi^{NS}_{\frac{1}{32}}+\chi^{NS}_{\frac{33}{32}})(\chi^{NS}_{\frac{5}{32}})\right|^2+\left|(\chi^{NS}_{\frac{1}{32}}+\chi^{NS}_{\frac{33}{32}})^3(\chi^{NS}_{\frac{5}{32}})\right|^2+\\
&+\frac{3}{2}\left|(\chi^R_{\frac{1}{16}}+\chi^R_{\frac{9}{16}})(\chi^{NS}_{\frac{1}{32}}+\chi^{NS}_{\frac{33}{32}})(\chi^R_{\frac{5}{16}}+\chi^R_{\frac{29}{16}})(\chi^{NS}_{\frac{5}{32}})\right|^2+\frac{3}{4}\left|(\chi^R_{\frac{1}{16}}+\chi^R_{\frac{9}{16}})^2(\chi^{NS}_{\frac{5}{32}})^2\right|^2\\
&+\frac{3}{4}\left|(\chi^{NS}_{\frac{1}{32}}+\chi^{NS}_{\frac{33}{32}})(\chi^R_{\frac{5}{16}}+\chi^R_{\frac{29}{16}})^2(\chi^{NS}_{\frac{5}{32}})\right|^2+3\left|(\chi^{NS}_{\frac{1}{32}}+\chi^{NS}_{\frac{33}{32}})^2(\chi^{NS}_{\frac{5}{32}})^2\right|^2+\\
&+\frac{3}{2}\left|(\chi^R_{\frac{1}{16}}+\chi^R_{\frac{9}{16}})(\chi^R_{\frac{5}{16}}+\chi^R_{\frac{29}{16}})(\chi^{NS}_{\frac{5}{32}})^2\right|^2+\frac{3}{4}\left|(\chi^R_{\frac{5}{16}}+\chi^R_{\frac{29}{16}})^2(\chi^{NS}_{\frac{5}{32}})^2\right|^2+\\
&+4\left|(\chi^{NS}_{\frac{1}{32}}+\chi^{NS}_{\frac{33}{32}})(\chi^{NS}_{\frac{5}{32}})^3\right|^2+2\left|(\chi^{NS}_{\frac{5}{32}})^4\right|^2.
\end{aligned}
\tag{6.7}
$$

Again, the expression satisfies level matching, Bose-Fermi degeneracy and absence of tachyons. Since all coefficient in (6.7) are of the form

$$
a_i = \frac{N}{128}\,, \qquad \text{with } N \in \mathbb{Z}_+ \tag{6.8}
$$

and taking into account that the Ramond ground states in $\chi^R_{\frac{1}{16}}$, $\chi^R_{\frac{9}{16}}$, $\chi^R_{\frac{5}{16}}$ and $\chi^R_{\frac{29}{16}}$ are twofold degenerate the coefficient 2^4 in (6.6) guarantees that the twisted sector partition function does indeed allow the interpretation as a trace over states in a Hilbert space. The lowest energy states in (6.7) are

$$
I(q,\overline{q}) = \frac{1}{8}\left|(\chi^{NS}_{\frac{1}{32}}+\chi^{NS}_{\frac{33}{32}})^4\right|^2 + \text{higher terms} \tag{6.9}
$$

leading to a massless twisted sector ground state. This gives rise to **2** additional chiral multiplets in the $\mathbb{Z}_2$ twisted sector. Thus, the overall massless spectrum of the $\frac{(6)^4\times S^1}{\sigma^*}$ model is

$$
\text{SUGRA} + (151_u + 2_t)\ \text{chiral multiplets}\,. \tag{6.10}
$$

In contrast to the $(3)^5$ Gepner model here the σ^* action has fixed points which give rise to new chiral multiplets in the twisted sector.

6.2 $(\mathbb{P}_{1,1,2,2,2}[8]\times S^1)/\mathbb{Z}_2$

The massless spectrum in the $\mathbb{Z}_2$ untwisted sector consists of the $\mathcal{N}=2$ supergravity multiplet in addition to 89 chiral multiplets. Taking the uncharged characters of the $k=2$ and $k=6$ unitary models and their behavior under the spectral flow into account we arrive at the following expression for the $\sigma^* \underset{1}{\square}$ partition function

$$\sigma^* \underset{1}{\square} = \frac{2}{|\eta|^2} \left| \sqrt{\frac{\eta}{\theta_2}} \right|^2 |O_1 V_1 - V_1 O_1|^2 \frac{1}{2} \sum_{i,j,k=0}^{1} \sum_{l,m=0}^{3} \Big| \psi^{2i}(\sigma^*) \psi^{2j}(\sigma^*) \psi^{2k}(\sigma^*) \Big(\chi^{2l}(\sigma^*) \chi^{2m}(\sigma^*) + \\ + \chi^{6-2l}(\sigma^*) \chi^{6-2m}(\sigma^*) \Big) \Big|^2 . \tag{6.11}$$

We have denoted the characters of the $k = 2$ unitary model by ψ^l and the characters of the $k = 6$ unitary model by χ^l. After a modular S-transformation we obtain for the twisted sector partition function

$$1 \underset{\sigma^*}{\square} = \frac{2^5}{|\eta|^2} \left| \sqrt{\frac{\eta}{\theta_4}} \right|^{12} |S_1(O_1 + V_1) - (O_1 + V_1) S_1|^2 \, I(q, \overline{q}) \, . \tag{6.12}$$

with

$$\begin{aligned} I(q,\overline{q}) = {} & \frac{1}{64} \left| (\psi_0 + \psi_{\frac{1}{2}})^3 (\chi^R_{\frac{1}{16}} + \chi^R_{\frac{9}{16}})^2 \right|^2 + \frac{1}{16} \left| (\psi_0 + \psi_{\frac{1}{2}})^3 (\chi^{NS}_{\frac{1}{32}} + \chi^{NS}_{\frac{33}{32}})^2 \right|^2 + \\ & + \frac{1}{32} \left| (\psi_0 + \psi_{\frac{1}{2}})^3 (\chi^R_{\frac{1}{16}} + \chi^R_{\frac{9}{16}})(\chi^R_{\frac{5}{16}} + \chi^R_{\frac{29}{16}}) \right|^2 + \frac{1}{64} \left| (\psi_0 + \psi_{\frac{1}{2}})^3 (\chi^R_{\frac{5}{16}} + \chi^R_{\frac{29}{16}})^2 \right|^2 + \\ & + \frac{1}{4} \left| (\psi_0 + \psi_{\frac{1}{2}})^3 (\chi^{NS}_{\frac{1}{32}} + \chi^{NS}_{\frac{33}{32}})(\chi^{NS}_{\frac{5}{32}}) \right|^2 + \frac{1}{4} \left| (\psi_0 + \psi_{\frac{1}{2}})^3 (\chi^{NS}_{\frac{5}{32}})^2 \right|^2 + \\ & + \frac{3}{32} \left| (\psi_0 + \psi_{\frac{1}{2}})^2 (\psi_{\frac{1}{16}}) (\chi^R_{\frac{1}{16}} + \chi^R_{\frac{9}{16}})^2 \right|^2 + \frac{3}{8} \left| (\psi_0 + \psi_{\frac{1}{2}})^2 (\psi_{\frac{1}{16}}) (\chi^{NS}_{\frac{1}{32}} + \chi^{NS}_{\frac{33}{32}})^2 \right|^2 + \\ & + \frac{3}{16} \left| (\psi_0 + \psi_{\frac{1}{2}})^2 (\psi_{\frac{1}{16}}) (\chi^R_{\frac{1}{16}} + \chi^R_{\frac{9}{16}})(\chi^R_{\frac{5}{16}} + \chi^R_{\frac{29}{16}}) \right|^2 + \frac{3}{32} \left| (\psi_0 + \psi_{\frac{1}{2}})^2 (\psi_{\frac{1}{16}}) (\chi^R_{\frac{5}{16}} + \chi^R_{\frac{29}{16}})^2 \right|^2 + \\ & + \frac{3}{2} \left| (\psi_0 + \psi_{\frac{1}{2}})^2 (\psi_{\frac{1}{16}}) (\chi^{NS}_{\frac{1}{32}} + \chi^{NS}_{\frac{33}{32}})(\chi^{NS}_{\frac{5}{32}}) \right|^2 + \frac{3}{2} \left| (\psi_0 + \psi_{\frac{1}{2}})^2 (\psi_{\frac{1}{16}}) (\chi^{NS}_{\frac{5}{32}})^2 \right|^2 + \\ & + \frac{3}{16} \left| (\psi_0 + \psi_{\frac{1}{2}}) (\psi_{\frac{1}{16}})^2 (\chi^R_{\frac{1}{16}} + \chi^R_{\frac{9}{16}})^2 \right|^2 + \frac{3}{4} \left| (\psi_0 + \psi_{\frac{1}{2}}) (\psi_{\frac{1}{16}})^2 (\chi^{NS}_{\frac{1}{32}} + \chi^{NS}_{\frac{33}{32}})^2 \right|^2 + \\ & + \frac{3}{8} \left| (\psi_0 + \psi_{\frac{1}{2}}) (\psi_{\frac{1}{16}})^2 (\chi^R_{\frac{1}{16}} + \chi^R_{\frac{9}{16}})(\chi^R_{\frac{5}{16}} + \chi^R_{\frac{29}{16}}) \right|^2 + \frac{3}{16} \left| (\psi_0 + \psi_{\frac{1}{2}}) (\psi_{\frac{1}{16}})^2 (\chi^R_{\frac{5}{16}} + \chi^R_{\frac{29}{16}})^2 \right|^2 + \\ & + 3 \left| (\psi_0 + \psi_{\frac{1}{2}}) (\psi_{\frac{1}{16}})^2 (\chi^{NS}_{\frac{1}{32}} + \chi^{NS}_{\frac{33}{32}})(\chi^{NS}_{\frac{5}{32}}) \right|^2 + 3 \left| (\psi_0 + \psi_{\frac{1}{2}}) (\psi_{\frac{1}{16}})^2 (\chi^{NS}_{\frac{5}{32}})^2 \right|^2 + \\ & + \frac{1}{8} \left| (\psi_{\frac{1}{16}})^3 (\chi^R_{\frac{1}{16}} + \chi^R_{\frac{9}{16}})^2 \right|^2 + \frac{1}{2} \left| (\psi_{\frac{1}{16}})^3 (\chi^{NS}_{\frac{1}{32}} + \chi^{NS}_{\frac{33}{32}})^2 \right|^2 + \\ & + \frac{1}{4} \left| (\psi_{\frac{1}{16}})^3 (\chi^R_{\frac{1}{16}} + \chi^R_{\frac{9}{16}})(\chi^R_{\frac{5}{16}} + \chi^R_{\frac{29}{16}}) \right|^2 + \frac{1}{8} \left| (\psi_{\frac{1}{16}})^3 (\chi^R_{\frac{5}{16}} + \chi^R_{\frac{29}{16}})^2 \right|^2 + \\ & + 2 \left| (\psi_{\frac{1}{16}})^3 (\chi^{NS}_{\frac{1}{32}} + \chi^{NS}_{\frac{33}{32}})(\chi^{NS}_{\frac{5}{32}}) \right|^2 + 2 \left| (\psi_{\frac{1}{16}})^3 (\chi^{NS}_{\frac{5}{32}})^2 \right|^2 . \end{aligned} \tag{6.13}$$

Again, one can check that this expression satisfies level matching, Bose-Fermi degeneracy, absence of tachyons and the allowance of an interpretation as a trace. The contribution from the internal sector starts like

$$I(q, \overline{q}) = \frac{1}{16} \left| \Big(\psi_0 + \psi_{\frac{1}{2}} \Big)^3 \ \Big(\chi^{NS}_{\frac{1}{32}} \Big)^2 \right|^2 + \text{higher terms} \tag{6.14}$$

and gives rise to **2** additional massless chiral multiplets in the $\mathbb{Z}_2$ twisted sector. Summarizing, the massless spectrum of the $\frac{(2)^3(6)^2 \times S^1}{\sigma^*}$ model is

$$\text{SUGRA} + (89_u + 2_t) \text{ chiral multiplets} \, . \tag{6.15}$$

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Finally, we would like to mention that we have also computed the $(1)^9$ and $(2)^6$ Gepner models. These models do not have a geometric phase, as formally $h_{11} = 0$. For the $(1)^9$ model we found **85** chiral multiplets in the untwisted sector and no massless states in the $\mathbb{Z}_2$ twisted sector. For the $(2)^6$ model we found **91** chiral multiplets in the untwisted sector and **2** additional massless states in the $\mathbb{Z}_2$ twisted sector.

7. Geometric interpretation

7.1 The geometry

The discussion is very similar to the case with the quintic. One new feature is the singular surface in the $\mathbb{P}_{11222}$ which must be blown up (One line in the moduli space of this model was investigated in [7]). Since the involution on the singular space extends to an involution on the resolved space this poses no problem. As in the previous case the Betti numbers of the quotient are

$$b_1 = 0\,, \qquad b_2 = 0\,, \qquad b_3 = h^{11} + h^{21} + 1\,. \tag{7.1}$$

The big difference is that the involution is free, that is it acts without fixed points. The reason is that the Fermat polynomials have only even degrees, and therefore no real solutions. For example the fixed point set in $\mathbb{P}_{11114}[8]$ is

$$\emptyset = \{x_1^8 + x_2^8 + x_3^8 + x_4^8 + x_5^2 = 0\} \in \mathbb{RP}_{11114} \tag{7.2}$$

Especially there should be no massless states in the twisted sector, unlike in the SCFT analysis.

In the remainder of this section we will discuss the topology in greater detail with the ultimate goal to show that $H_2(X;\mathbb{Z}) \neq \emptyset$. Unfortunately we will encounter technical difficulties and our calculation works only for $h^{11} > 1$, but we expect it to hold in general. This result is crucial for the physical explanation that we will offer in the next section.

7.2 Fundamental group and holonomy

The following discussion is valid for all $X = (Y \times S^1)/\mathbb{Z}_2$ that satisfy the following two conditions:

1. Y is a simply connected Calabi-Yau manifold (e.g. a toric hypersurface) with Hodge numbers h^{11}, h^{21}.
2. The $\mathbb{Z}_2$-action is free and inverts all classes in $H^{1,1}(Y)$.

The long exact homotopy sequence of the $\mathbb{Z}_2$-bundle $Y \times S^1 \to X$

$$\cdots \to \underbrace{\pi_1(\mathbb{Z}_2)}_{=0} \to \underbrace{\pi_1(Y \times S^1)}_{=\mathbb{Z}} \to \pi_1(X) \to \underbrace{\pi_0(\mathbb{Z}_2)}_{=\mathbb{Z}_2} \to \underbrace{\pi_0(Y \times S^1)}_{=0} \to \cdots \tag{7.3}$$

implies that $\pi_1(X)$ must be infinite (Indeed $\pi_1(X) = \mathbb{Z}_2 * \mathbb{Z}_2$). But remember that for any compact manifold with torsion free G_2-structure (i.e. the closed and coclosed 3-form) the following is equivalent: $|\pi_1| < \infty \Leftrightarrow \mathrm{Hol} = G_2$. Since nevertheless $b_1 = 0$ we conclude that

$$\mathrm{SU}(3) \subsetneq \mathrm{Hol} \subsetneq G_2\,. \tag{7.4}$$

7.3 The Cartan-Leray spectral sequence

One of the most useful tools to compute the homology of free quotients of arbitrary spaces is the Cartan-Leray spectral sequence (see [36] for a similar application). However it turns out that there is an ambiguity which we cannot resolve. The result will be that $H_2(X;\mathbb{Z}) \neq 0$ if $h^{11} > 1$, and undetermined otherwise.

A spectral sequence is a systematic scheme to compute (co)homology groups from other (hopefully more accessible) data. This particular one is valid for arbitrary spaces (we choose $Y \times S^1$) with free (proper) $\mathbb{Z}_2$-action (of course it is valid for more general groups but this suffices for our purposes). It starts at $E^2_{p,q} = \hat{H}_p(\mathbb{Z}_2, H_q(Y \times S^1, \mathbb{Z}))$ and converges to $H_{p+q}(X;\mathbb{Z})$, which is exactly what we are interested in.

In this sequence only non-negative p, q have $E^2_{p,q} \neq 0$ so we can draw the initial data in the first quadrant. We do not assume that the reader is familiar with group cohomology $\hat{H}_p(\mathbb{Z}_2, F)$ — F denotes some $\mathbb{Z}_2$-module, that is comes with an $\mathbb{Z}_2$-action like in our case the homology groups $H_q(Y \times S^1, \mathbb{Z})$. So let us quote the fundamental results:

- $\hat{H}_p(\mathbb{Z}_2, \mathbb{Z}) = H_p(\mathbb{RP}_\infty)$ (where $\mathbb{Z}$ has the trivial $\mathbb{Z}_2$-action)
- $\hat{H}_p(\mathbb{Z}_2, \widetilde{\mathbb{Z}}) = H_p(\mathbb{RP}_\infty; \widetilde{\mathbb{Z}}) = \{\mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \ldots$ where $\widetilde{\mathbb{Z}}$ are the integers with the non-trivial $\mathbb{Z}_2$-action
- $\hat{H}_p(\mathbb{Z}_2, 0) = 0$
- $\hat{H}_0(\mathbb{Z}_2, F) = F \big/ \langle x - gx \rangle$ where gx denotes the generator of $\mathbb{Z}_2$ acting on $x \in F$.

For example we have (by assumption)

$$\begin{aligned} gx = -x \qquad & \forall x \in H_2(Y \times S^1; \mathbb{Z}) \\ & \Rightarrow E^2_{0,2} = \hat{H}_0(\mathbb{Z}_2, H_2(Y \times S^1; \mathbb{Z})) = (\mathbb{Z}/2\mathbb{Z})^{h^{11}} = \mathbb{Z}_2^{h^{11}} \end{aligned} \tag{7.5}$$

So far we can evaluate

$$E^2_{p,q}(X) = \begin{array}{c|cccc} \uparrow & \mathbb{Z}_2^{h^{11}} & 0 & \mathbb{Z}_2^{h^{11}} & 0 \\ q & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 \\ & \mathbb{Z} & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 \\ \hline & & & p & \rightarrow \end{array} \tag{7.6}$$

Now starts the real work: At each entry in $E^2_{p,q}$ there is a map that goes up $2-1$ and 2 to the left. The cohomology at each point is then $E^3_{p,q}$ and there are more maps, this time $3-1$ up and 3 left. This continues ("converges") to $E^\infty_{p,q}$, whose $p+q = \text{const}$ diagonals are then the "associated graded complex" for $H_{p+q}(X;\mathbb{Z})$. The diagonal is empty if and only if the cohomology $H_{p+q}(X;\mathbb{Z})$ vanishes.

The only things that possibly influence the $p+q=2$ diagonal will be the maps $d_2 : E^2_{2,1} \to E^2_{0,2}$ and $d_3 : E^3_{3,0} \to E^3_{0,2}$. Depending on the first map either $E^3_{p,q} = \mathbb{Z}_2^{h^{11}} / \operatorname{img} d_2 = \mathbb{Z}_2^{h^{11}}$ or $\mathbb{Z}_2^{h^{11}-1}$. The second map d_3 can in principle also kill another $\mathbb{Z}_2$. Since we must assume the worst case this argument only shows: If $h^{11} > 2$ then $H_2(X;\mathbb{Z}) \neq \emptyset$.

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7.4 The $h^{11} = 2$ case

We can strengthen the result a little bit by further investigating the topology of X. Via the projection on the first factor in $Y \times S^1$ we see that X is a S^1-bundle over $Y/\mathbb{Z}_2$. The topology of the bundle is fixed by the fact that X is orientable and $Y/\mathbb{Z}_2$ is not. So we can also determine the (co)homology of X by first calculating the homology of $Y/\mathbb{Z}_2$ with the Cartan-Leray spectral sequence and then use Leray's Theorem to compute the cohomology of the bundle from the cohomology of the base and the fiber.

The Cartan-Leray spectral sequence for $Y/\mathbb{Z}_2$ starts with

$$E^2_{p,q}(Y/\mathbb{Z}_2) = \begin{array}{c|cccc} \uparrow & \mathbb{Z}_2^{h^{11}} & 0 & \mathbb{Z}_2^{h^{11}} & 0 \\ q & 0 & 0 & 0 & 0 \\ & \mathbb{Z} & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 \\ \hline & & & p & \rightarrow \end{array} \tag{7.7}$$

Now the ambiguity is reduced to $H_2(Y/\mathbb{Z}_2;\mathbb{Z}) = \mathbb{Z}_2^{h^{11}}$ or $\mathbb{Z}_2^{h^{11}-1}$. In fact the only consistent possibilities are

$$H_p(Y/\mathbb{Z}_2;\mathbb{Z}) = \begin{cases} 0 & p=6 \\ \mathbb{Z}_2 & p=5 \\ \mathbb{Z}^{h^{11}} & p=4 \\ \mathbb{Z}^{h^{21}+1}[\oplus\mathbb{Z}_2] & p=3 \\ \mathbb{Z}_2^{h^{11}-1}[\oplus\mathbb{Z}_2] & p=2 \\ \mathbb{Z}_2 & p=1 \\ \mathbb{Z} & p=0 \end{cases} \tag{7.8}$$

Then Leray's Theorem amounts to a spectral sequence (this time for cohomology) with $E_2^{p,q} = H^p(Y/\mathbb{Z}_2;\mathcal{H}^q(S^1;\mathbb{Z}))$. We find

$$E_2^{p,q}(X) = \begin{array}{c|ccccccc} \uparrow q & 0 & \mathbb{Z}_2 & \mathbb{Z}^{h^{11}} & \mathbb{Z}^{h^{21}+1}[\oplus\mathbb{Z}_2] & \mathbb{Z}_2^{h^{11}-1}[\oplus\mathbb{Z}_2] & \mathbb{Z}_2 & \mathbb{Z} \\ & \mathbb{Z} & 0 & \mathbb{Z}_2 & \mathbb{Z}^{h^{21}+1}\oplus\mathbb{Z}_2^{h^{11}-1}[\oplus\mathbb{Z}_2] & \mathbb{Z}^{h^{11}}[\oplus\mathbb{Z}_2] & 0 & \mathbb{Z}_2 \\ \hline & & & & p \rightarrow & & & \end{array} \tag{7.9}$$

The only interesting differential is $d_2 : E_2^{1,1} \to E_2^{3,0}$ and it has to vanish because the $\mathbb{Z}_2$ at $(p,q) = (1,1)$ has to survive as $H^2(X;\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. But then everything survives to E_∞ and we get

$$H^3(X;\mathbb{Z})_{\text{tor}} = H_2(X;\mathbb{Z})_{\text{tor}} = \mathbb{Z}_2^{h^{11}-1} \quad \text{or} \quad \mathbb{Z}_2^{h^{11}}. \tag{7.10}$$

8. Resolution of the puzzle

We have seen that the SCFT and the geometric analysis never yield the same result. The resolution to this puzzle is that, as derived from mirror symmetry [47, 48], the Gepner model corresponds to a point in the complexified Kähler moduli space where a non-trivial background NS-NS two form flux, B, has been turned on. For instance, for the quintic Calabi-Yau it was shown that the complex Kähler parameter of the Gepner model is

$$B + iJ = \frac{1}{2} + \frac{i}{2}\cot\left(\frac{\pi}{5}\right). \tag{8.1}$$

Under the action of σ^* the 2-cycle which is Poincare dual to the Kähler form, J, combines with the 1-cycle of the additional S^1 to a 3-cycle and gives rise to one chiral multiplet in three dimensions. The two-form, B, however is mapped to $-B$ under the action of σ^* and is therefore projected out. Thus, the two-form is not any longer a continuous parameter. Due to the fact that B is only defined modulo one, there actually exist two allowed discrete values for the background B field, $B = 0, 1/2$. This is very similar to what happens for compact type-I models where also the NS-NS two-form does not survive the orientifold projection but nevertheless gives rise to disconnected branches of the moduli space [49].

To summarize, the supergravity point lies on the $B = 0$ branch whereas the Gepner point lies in the $B = 1/2$ branch of the G_2 moduli space. Thus it is no surprise that the SCFT computation and the supergravity computation yield different results.

Does this branch with $B = 1/2$ belong to the M-theory moduli space in four dimensions, as well? The answer is no. In the three dimensional M-theory compactification on the G_2-manifold times a circle the NS-NS two form flux on the G_2-manifold is lifted to a three form flux, where two components of C_{ijk} lie on the seven dimensional G_2 manifold and one component on the circle. Therefore, decompactifying this model to four dimensions by making the circle very big, one looses the non-trivial three form flux. This is consistent with the result derived in [50], that in M-theory on a Calabi-Yau manifold the non-geometric phases are absent.

In the case of the quintic the geometric picture is not clear because we do not understand how to handle the singularities. But for free involutions the quotient is a genuine manifold and the discrete moduli must be visible at large volume. And indeed we have seen that $H_2(X;\mathbb{Z})_{\rm tor} = H^3(X;\mathbb{Z})_{\rm tor} \neq \emptyset$, at least if $h^{11} > 1$. So there is the possibility for a flat but nontrivial B-field, that is one with a characteristic class in $H^3(X;\mathbb{Z})$ that is pure torsion. With other words there is the possibility of turning on discrete $\mathbb{Z}_2$ two form flux through the nontrivial 2-cycles.

9. Conclusions

In this paper we have presented a class of SCFTs describing certain points in the moduli space of G_2 compactifications of type-II strings. These models are given by anti-holomorphic quotients of $c = 9$ Gepner models times a circle. We have identified the anti-holomorphic involution in the $c = 9$ SCFT simply as the operation of charge conjugation in each tensor factor. However, it turned out that the precise action of this charge conjugation on all states in the Hilbert space, in particular on the neutral ones, is not that straightforwardly to determine. Instead for certain simple models we were able to distinguish the σ^* even from the σ^* odd states by intelligent guesswork. Of course it would be a big advance to find the general solution to this pure CFT problem, which involves determining the representation theory of a $\mathcal{W}(2,4,6,8,10)$ algebra. This would allow one, using the general construction given in this paper, to compute quite a number of exactly solvable points on various G_2 manifolds.

Here we have considered three particular SCFTs in quite some detail including a quotient involving the quintic Calabi-Yau. In all cases considered in turned out that the SCFT result for the twisted sector massless spectrum is different from the supergravity result. In was pointed out that this is actually no surprise taking into account that the Gepner model corresponds to a point in the non-linear sigma model moduli space where a background two-form flux has been turned on. In the G_2 model this modulus is frozen to discrete values so that the SCFT and the naive large volume geometry lie on separate branches of the moduli space. We also pointed out that the Gepner model branch is absent in the corresponding four dimensional M-theory compactification.

Being equipped with a class of exactly solvable SCFTs one can now move forward and investigate issues like the behavior of the model under deformations away from the exactly solvable point or the generation of a superpotential. Moreover, one could try using the abstract boundary state [34, 35, 36] formalism to find stable D-branes in the deep stringy regime of G_2 compactifications.

On the one hand one might consider simple generalizations of the construction presented in this paper, but on the other hand one might also contemplate completely different conformal field theory constructions. which are not of the form of a toroidal orbifold or an anti-holomorphic quotient of a Calabi-Yau times a circle. Analogous to [51] one might start with tensor products of the $\mathcal{N}=1$ unitary models from the very beginning, even though it seems to be quite hard to implement a spectral flow in such models.

It is known that Gepner models are special points in the Landau-Ginzburg phase of the corresponding $(2,2)$ linear sigma model. Thus, it would be interesting to see whether one can treat these anti-holomorphic quotients directly in the Landau-Ginzburg model. It would also be interesting to generalize the construction of superconformal field theories described in this paper to the case of eight dimensional manifolds with Spin(7) holonomy.

Acknowledgments

We would like to thank Anamaria Font, Boris Körs and Dieter Lüst for discussion and Radu Tatar for comments about the manuscript. The group is supported in part by the EEC contract ERBFMRXCT96-0045.

A. Unitary HWR of the $\mathcal{N}=0$ Virasoro algebra

We summarize some basis data about the $\mathcal{N}=0$ super Virasoro algebra. The central charge of the unitary, rational models is

$$c = 1 - \frac{6}{m(m+1)}\,, \qquad m = 3,4,5,\ldots \tag{A.1}$$

and the HWR are

$$h^m_{r,s} = \frac{\left((m+1)r - ms\right)^2 - 1}{4m(m+1)}\,, \qquad 1 \le r \le m-1, 1 \le s \le m\,. \tag{A.2}$$

Moreover, one has the reflection symmetry $h^m_{r,s} = h^m_{m-r,m+1-s}$. The characters in the NS and R sector can be expressed as

$$\chi_{r,s} = \frac{1}{\eta(\tau)}\left(\Theta_{(m+1)r-ms,m(m+1)}(\tau) - \Theta_{(m+1)r+ms,m(m+1)}(\tau)\right). \tag{A.3}$$

3	$\frac{7}{5}$	$\frac{2}{5}$	0
$\frac{13}{8}$	$\frac{21}{40}$	$\frac{1}{40}$	$\frac{1}{8}$
$\frac{2}{3}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{3}$
$\frac{1}{8}$	$\frac{1}{40}$	$\frac{21}{40}$	$\frac{13}{8}$
0	$\frac{2}{5}$	$\frac{7}{5}$	3

Table 1: Conformal grid for $m = 5$.

3	$\frac{29}{16}$	$\frac{5}{6}$	$\frac{5}{16}$	0
$\frac{67}{32}$	$\frac{33}{32}$	$\frac{41}{96}$	$\frac{1}{32}$	$\frac{3}{32}$
$\frac{5}{4}$	$\frac{9}{16}$	$\frac{1}{12}$	$\frac{1}{16}$	$\frac{1}{4}$
$\frac{23}{32}$	$\frac{5}{32}$	$\frac{5}{96}$	$\frac{5}{32}$	$\frac{23}{32}$
$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{12}$	$\frac{9}{16}$	$\frac{5}{4}$
$\frac{3}{32}$	$\frac{1}{32}$	$\frac{41}{96}$	$\frac{33}{32}$	$\frac{67}{32}$
0	$\frac{5}{16}$	$\frac{5}{6}$	$\frac{29}{16}$	3

Table 2: Superconformal grid for $m = 6$.

For the modular S-matrix one gets

$$S_{r_1,s_1;r_2,s_2} = \sqrt{\frac{8}{m(m+1)}}(-1)^{(r_1+s_1)(r_2+s_2)} \sin\left(\frac{\pi r_1 r_2}{m+1}\right) \sin\left(\frac{\pi s_1 s_2}{m+1}\right). \tag{A.4}$$

For $m = 5$ one obtains the conformal grid shown in table 1.

B. Unitary HWR of the $\mathcal{N} = 1$ super Virasoro algebra

We summarize some basis data about the $\mathcal{N} = 1$ super Virasoro algebra. The central charge of the unitary, rational models is

$$c = \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right), \qquad m = 3, 4, 5, \dots \tag{B.1}$$

and the HWR are

$$h^m_{r,s} = \frac{((m+2)r - ms)^2 - 4}{8m(m+2)} + \frac{1-(-1)^{r-s}}{32}, \qquad 1 \le r \le m-1\,, 1 \le s \le m+1 \tag{B.2}$$

where $r + s =$ even is the NS sector and $r + s =$ odd the R-sector. Moreover, one has the reflection symmetry $h^m_{r,s} = h^m_{m-r,m+2-s}$. The characters in the NS and R sector can be expressed as

$$\begin{aligned}
\chi^{NS}_{r,s} &= \frac{1}{\eta(\tau)}\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}}\left(\Theta_{(m+2)r-ms,m(m+2)}\left(\frac{\tau}{2}\right) - \Theta_{(m+2)r+ms,m(m+2)}\left(\frac{\tau}{2}\right)\right) \\
\chi^{R}_{r,s} &= \left(2 - \delta_{r,\frac{m}{2}}\delta_{s,\frac{m+2}{2}}\right)\frac{1}{\eta(\tau)}\sqrt{\frac{\theta_2(\tau)}{2\eta(\tau)}}\left(\Theta_{(m+2)r-ms,m(m+2)}\left(\frac{\tau}{2}\right) - \right. \\
&\qquad \left. - \Theta_{(m+2)r+ms,m(m+2)}\left(\frac{\tau}{2}\right)\right).
\end{aligned} \tag{B.3}$$

For the modular S-matrix one gets [52]

$$\begin{aligned}
S^{NS,NS}_{r_1,s_1;r_2,s_2} &= \frac{2}{\sqrt{m(m+2)}}\left(\cos\left(\frac{2\pi\lambda_1\lambda_2}{4m(m+2)}\right) - \cos\left(\frac{2\pi\lambda_1\overline{\lambda}_2}{4m(m+2)}\right)\right) \\
S^{\widetilde{NS},R}_{r_1,s_1;r_2,s_2} &= \frac{2}{\sqrt{2m(m+2)}}\left(\cos\left(\frac{2\pi\lambda_1\lambda_2}{4m(m+2)}\right) - (-1)^{r_1 s_1}\cos\left(\frac{2\pi\overline{\lambda}_1\lambda_2}{4m(m+2)}\right)\right)
\end{aligned} \tag{B.4}$$

with $\lambda_i = (m+2)r_i - ms_i$ and $\overline{\lambda}_i = (m+2)r_i + ms_i$. The conformal for the $m = 6$ unitary model is shown in table 2.

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